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LETTER TO THE EDITOR

The quantum multiboson algebra and generalized coherent states of $SU_q(1, 1)$

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Abstract. We give a generalization of the deformed oscillator and show that it satisfies the quantum Weyl–Heisenberg group. The nonlinear transformations from the usual deformed oscillator operators to the deformed k-boson operators form an Abelian group. The generalized Holstein–Primakoff realizations in terms of these multiboson operators are used to define the quantum group-theoretic coherent states of $SU_q(1, 1)$.

Recently the Jaynes–Cummings model (JCM) [1–4] which possesses quantum group [5, 6] dynamical symmetry was suggested [7, 8]. The dynamical algebra of the deformed JC Hamiltonian system is $SU_q(1, 1)\oplus SU(2)$. The dynamical properties of the system with respect to the Biedenharn–Macfarlane coherent states (BMCS) [9, 10] are investigated and showed that the periodic revivals of the generalized JCM are destroyed increasingly for larger values of the deformation parameter q. Another noticeable property of the system is the squeezing of the BMCS [11, 12].

In fact this system is a special case ($k=1$) of a more general system of a two-level atom interacting with deformed k-bosons. The general form of the system consists of a two-level atom interacting with deformed k-bosons [13, 14] with generalized Holstein–Primakoff (HP) realizations and quantum group-theoretic coherent states of $SU_q(1, 1)$. It is interesting to probe the dynamical properties of the general interacting system.

In this paper, we discuss the quantum multiboson algebra and the properties of the quantum group-theoretic coherent states. It is well known that the deformed oscillator satisfies the quantum Weyl–Heisenberg algebra $H_q(4)$

$$[a_q, a_q^\dagger]=[N+1]-[N] \qquad [N, a_q]=-a_q \qquad [N, a_q^\dagger]=a_q^\dagger. \tag{1}$$

Now, we give a more general description of the quantum Weyl–Heisenberg group, the k-boson realization. First of all, we introduce the nonlinear transformations

$$\begin{aligned} A_{(k,q)}^\dagger &\equiv \left(\frac{[N-k]!}{[N]!}\left[\left\langle\frac{N}{k}\right\rangle\right]\right)^{1/2}(a_q^\dagger)^k \\ A_{(k,q)} &\equiv (a_q)^k\left(\left[\left\langle\frac{N}{k}\right\rangle\right]\frac{[N-k]!}{[N]!}\right)^{1/2} \\ N_{(k,q)} &\equiv A_{(k,q)}^\dagger A_{(k,q)} \end{aligned} \tag{2}$$

where k is a positive integer and $\langle x\rangle$ is defined as the greatest integer less than or equal to x, functions of the number operator $N=a^\dagger a$ are only evaluated on eigenstates of

N, and assume the value of the functions of the respective eigenvalues. It is not difficult to check that the above-defined deformed k-boson operators $A_{(k,q)}$ and $A^{\dagger}_{(k,q)}$ also satisfy the quantum Weyl–Heisenberg group $\mathrm{H}_q(4)$

$$
\begin{aligned}
&[A_{(k,q)}, A^{\dagger}_{(K,q)}] = \left[\left\langle\frac{N}{k}\right\rangle + 1\right] - \left[\left\langle\frac{N}{k}\right\rangle\right] \\
&\left[\left\langle\frac{N}{k}\right\rangle, A^{\dagger}_{(k,q)}\right] = A^{\dagger}_{(k,q)} \\
&\left[\left\langle\frac{N}{k}\right\rangle, A_{(k,q)}\right] = -A_{(k,q)}.
\end{aligned}
\tag{3}
$$

The nonlinear transformations $\{F_{(k)}, k \text{ is integer}\}$, $F_{(k)}: a^{\dagger}_q \to F_{(k)}(a^{\dagger}_q) \equiv A^{\dagger}_{(k,q)}$ form a semigroup. If we envisage a situation in which we wish to compute general moments of quantities such as

$$
X_{(k,q)} \equiv \frac{1}{\sqrt{2}}(A_{(k',q)} + A^{\dagger}_{(k',q)}) \qquad P_{(k,q)} \equiv \frac{1}{\sqrt{2}}(A_{(k',q)} - A^{\dagger}_{(k',q)}) \tag{4}
$$

in eigenstates of the number operator $N_{(k,q)}$, associated with generalized k' and k deformed bosons, respectively, then we are led to consider the expectation

$$
\langle km|(A^{\dagger}_{(k',q)})^u (A_{(k',q)})^v|km'\rangle
$$

where k, k', u, v, m, m' are positive integers. It is a straightforward exercise to evaluate this expectation

$$
\langle km|(A^{\dagger}_{(k',q)})^u (A_{(k',q)})^v|km'\rangle = \left(\frac{\left[\left\langle\frac{km}{k'}\right\rangle\right]^{*}!\left[\left\langle\frac{km'}{k'}\right\rangle\right]!}{\left[\left\langle\frac{km}{k'} - u\right\rangle\right]^{*}!\left[\left\langle\frac{km'}{k'} - v\right\rangle\right]!}\right)^{1/2} \delta_{m,m'+l} \tag{5}
$$

where we have defined $t \equiv (k'/k)(u - v)$. It should be noticed that when $u = v$, then $t = 0$ and the expectation (5) always has non-zero values (for $m = m'$). When $u \neq v$, (5) vanishes unless t is an integer. The expectation (5) depends only on k' and k through their ratio $r = k'/k$. Here r is the positive rational fraction of the fractional transformation $F_{(r)}$. We may equate $\langle km|(A^{\dagger}_{(k',q)})^u (A_{(k',q)})^v|km'\rangle$ formally to an expectation involving fractional photons

$$
\langle km|(A^{\dagger}_{(k',q)})^u (A_{(k',q)})^v|km'\rangle = \langle m|(A^{\dagger}_{(r,q)})^u (A_{(r,q)})^v|m'\rangle. \tag{6}
$$

Notice the properties of the nonlinear transformation $F_{(k)}$

$$
F_{(1)}(a^{\dagger}_q) = a^{\dagger}_q \tag{7}
$$

and

$$
F_{(k)} \circ F_{(k')}(a^{\dagger}_q) = F_{(kk')}(a^{\dagger}_q) \tag{8}
$$

if we define the inverse transformation $F^{-1}_{(k)}$ as

$$
F^{-1}_{(k)} \circ F_{(k)}(a^{\dagger}_q) = a^{\dagger}_q = F_{(1)}(a^{\dagger}_q) \tag{9}
$$

we may equate $F^{-1}_{(k)} = F_{(1/k)}$. Similarly we have

$$
F^{-1}_{(k)} \circ F_{(k')} = F_{(k'/k)} = F_{(r)} \tag{10}
$$

where $r \equiv k'/k$ is a positive rational number. It is this extension which permits us to define fractional photons. Now we have extended the semigroup of the nonlinear transformation $F_{(k)}$ to an Abelian group $\{F_{(k)}$: rational $k > 0\}$.

Using the deformed k-boson operators, we can realize the quantum algebra $SU_q(1,1)$ by the following HP relations

$$\begin{aligned} J_+^{(k)} &= A^\dagger_{(k,q)}\left[2\sigma+\left\langle\frac{N}{k}\right\rangle\right]^{1/2} \\ J_-^{(k)} &= \left[2\sigma+\left\langle\frac{N}{k}\right\rangle\right]^{1/2} A_{(k,q)} \\ J_3^{(k)} &= \sigma+\left\langle\frac{N}{k}\right\rangle. \end{aligned} \tag{11}$$

The operators $J_\pm^{(k)}$ and $J_3^{(k)}$ satisfy the commutation relations of quantum algebra $SU_q(1,1)$

$$[J_3^{(k)}, J_\pm^{(k)}] = \pm J_\pm^{(k)} \qquad [J_+^{(k)}, J_-^{(k)}] = -[2J_3^{(k)}]. \tag{12}$$

The representation of the $SU_q(1,1)$ quantum algebra is spanned by the states $|0\rangle, |k\rangle, |2k\rangle, \ldots$.

The quantum group-theoretic coherent states of $SU_q(1,1)$ are defined as

$$|\sigma, k, \alpha\rangle \equiv \frac{1}{Q}\exp(\alpha J_+^{(k)})|0\rangle \tag{13}$$

where $\alpha \in C$ and Q is the renormalization coefficient. Expanding the exponential we obtained

$$|\sigma, k, \alpha\rangle = \frac{1}{Q}\sum_{l=0}^{\infty}\frac{\alpha^l}{l!}(J_+^{(k)})^l|0\rangle = \frac{1}{Q}\sum_{l=0}^{\infty}\frac{\alpha^l}{l!}\left(\frac{[2\sigma+l-1]![l]!}{[2\sigma-1]!}\right)^{1/2}|kl\rangle. \tag{14}$$

This is the general expression for the quantum group-theoretic coherent states of $SU_q(1,1)$. The inner product for the coherent state is

$$\langle\sigma, k, \alpha|\sigma, k, \alpha\rangle = \frac{1}{Q^2}\sum_{l=0}^{\infty}\frac{|\alpha|^{2l}}{(l!)^2}\left|\frac{[2\sigma+l-1]![l]!}{[2\sigma-1]!}\right|. \tag{15}$$

It follows that

$$Q = \frac{1}{\sqrt{\sum_{l=0}^{\infty}\frac{|\alpha|^{2l}}{(l!)^2}\left|\frac{[2\sigma-l+1]![l]!}{[2\sigma-1]!}\right|}}. \tag{16}$$

All required moments may be obtained from evaluation of the expectation

$$\begin{aligned} &\langle\sigma, k, \alpha|(A^\dagger_{(k',q)})^u(A_{(k',q)})^v|\sigma, k, \alpha\rangle \\ &= \frac{1}{Q^2}\sum_{m,m'=0}^{\infty}\frac{\bar{\alpha}^m\alpha^{m'}}{m!\,m'!}\langle km|(A^\dagger_{(k',q)})^u(A_{(k',q)})^v|km'\rangle \\ &\quad\times\left(\frac{[2\sigma+m-1]^*![m]^*!}{[2\sigma-1]^*!}\,\frac{[2\sigma+m'-1]![m']!}{[2\sigma-1]}\right)^{1/2} \\ &= \frac{1}{Q^2}\bar{\alpha}^t\sum_{m=0}^{\infty}\frac{|\alpha|^{2m}}{(m+t)!\,m!}\left(\frac{\left[\left\langle\frac{m+t}{r}\right\rangle\right]^*!\left[\left\langle\frac{m}{r}\right\rangle\right]!}{\left[\left\langle\frac{m+t}{r}-u\right\rangle\right]^*!\left[\left\langle\frac{m}{r}-v\right\rangle\right]!}\right)^{1/2} \\ &\quad\times\left(\frac{[2\sigma+m+t-1]^*![m+t]^*!}{[2\sigma-1]^*!}\,\frac{[2\sigma+m-1]![m]!}{[2\sigma-1]!}\right)^{1/2}. \end{aligned} \tag{17}$$

To evaluate $\Delta X_{(k',q)}, \Delta P_{(k',q)}$ we use the expressions

$$\begin{aligned}(\Delta X_{(k',q)})^2 &= \frac{[\langle N/k'\rangle+1]-[\langle N/k'\rangle]}{2}+\langle A^\dagger_{(k',q)}A_{(k',q)}\rangle-|\langle A^\dagger_{(k',q)}\rangle|^2\\ &\quad+\mathrm{Re}(\langle (A^\dagger_{(k',q)})^2\rangle-\langle A^\dagger_{(k',q)}\rangle^2)\\ (\Delta P_{(k',q)})^2 &= \frac{[\langle (N/k')+1\rangle]-[\langle N/k'\rangle]}{2}+\langle A^\dagger_{(k',q)}A_{(k',q)}\rangle-|\langle A^\dagger_{(k',q)}\rangle|^2\\ &\quad-\mathrm{Re}(\langle (A^\dagger_{(k',q)})^2\rangle-\langle A^\dagger_{(k',q)}\rangle^2)\end{aligned} \tag{18}$$

where all expectations are taken with respect to the quantum group-theoretic coherent states of $\mathrm{SU}_q(1,1), |\sigma, k, \alpha\rangle$. From the general form of expectations we can obtain the moments which are necessary to evaluate the uncertainty $\Delta X_{(k',q)}$ and $\Delta P_{(k',q)}$

$$\begin{aligned}\langle A^\dagger_{(k',q)}\rangle &= \frac{1}{Q^2}\bar\alpha^r\sum_{m=0}^{\infty}\frac{|\alpha|^{2m}}{(m+r)!\,m!}\\ &\quad\times\left(\frac{[2\sigma+m+r-1]^*![m+r]^*!}{[2\sigma-1]^*!}\,\frac{[2\sigma+m-1]![m]!}{[2\sigma-1]!}\right)^{1/2}\\ &\quad\times\sqrt{\left[\left\langle\frac{m}{r}+1\right\rangle\right]^*}\qquad \text{if } r=\text{integer}\\ \langle A^\dagger_{(k',q)}\rangle &= 0\qquad\text{otherwise}\\ \langle A^\dagger_{(k',q)}A_{(k',q)}\rangle &= \frac{1}{Q^2}\sum_{m=0}^{\infty}\frac{|\alpha|^{2m}}{(m!)^2}\left|\frac{[2\sigma+m-1]![m]!}{[2\sigma-1]!}\right|\left|\left[\left\langle\frac{m}{r}\right\rangle\right]\right|\\ \langle (A^\dagger_{(k',q)})^2\rangle &= \frac{1}{Q^2}\bar\alpha^{2r}\sum_{m=0}^{\infty}\frac{|\alpha|^{2m}}{(m+2r)!\,m!}\\ &\quad\times\left(\frac{[2\sigma+m+2r-1]^*![m+2r]^*!}{[2\sigma-1]^*!}\,\frac{[2\sigma+m-1]![m]!}{[2\sigma-1]!}\right)^{1/2}\\ &\quad\times\sqrt{\left[\left\langle\frac{m}{r}+2\right\rangle\right]^*\left[\left\langle\frac{m}{r}+1\right\rangle\right]^*}\qquad\text{if } 2r=\text{integer}\\ \langle (A^\dagger_{(k',q)})^2\rangle &= 0\qquad\text{otherwise.}\end{aligned} \tag{19}$$

When $k'=1$, the following formulae can be checked. For $k=1$ we obtain

$$\begin{aligned}\langle a^\dagger_q\rangle &= \frac{1}{Q^2}\bar\alpha\sum_{l=0}^{\infty}\frac{|\alpha|^{2l}}{(l+1)!\,l!}\sqrt{[2\sigma+l]^*}\left|\frac{[2\sigma+l-1]![l+1]!}{[2\sigma-1]!}\right|\\ \langle (a^\dagger_q)^2\rangle &= \frac{1}{Q^2}\bar\alpha^2\sum_{l=0}^{\infty}\frac{|\alpha|^{2l}}{(l+2)!\,l!}\sqrt{[2\sigma+l]^*[2\sigma+l+1]^*}\left|\frac{[2\sigma+l-1]![l+1]!}{[2\sigma-1]!}\right|.\end{aligned} \tag{20}$$

For $k=2$,

$$\begin{aligned}\langle (a^\dagger_q)^2\rangle &= \frac{1}{Q^2}\bar\alpha\sum_{l=0}^{\infty}\frac{|\alpha|^{2l}}{(l+1)!\,l!}\sqrt{[2\sigma+l]^*[l+1]^*[2l+1][2l+2]}\\ &\quad\times\left|\frac{[2\sigma+l-1]![l+1]!}{[2\sigma-1]!}\right|.\end{aligned} \tag{21}$$

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References

[1] Jaynes E T and Cummings F W 1963 *Proc. IEEE* **51** 89
[2] Buck B and Sukumar C V 1981 *Phys. Lett.* **81A** 132
[3] Buzek V 1989 *Phys. Rev.* A **39** 3196
[4] Meschede D, Waltner H and Muller G 1985 *Phys. Rev. Lett.* **54** 551
[5] Drinfel'd V G 1986 Quantum Groups *Proc. ICM (Berkeley)* ed A M Gleason (Providence, RI: American Mathematical Society) p 798
[6] Jimbo M 1985 *Lett. Math. Phys.* **10** 63; 1986 *Commun. Math. Phys.* **102** 537
[7] Chaichian M, Ellinas D and Kulish P 1990 *Phys. Rev. Lett.* **65** 980
[8] Chang Z 1992 The time evolution operator of an atom in a deformed intense field *J. Phys. A: Math. Gen.* **25** L529; 1992 Dynamics of generalized photonic fields interacting with atoms *Phys. Rev.* A in press
[9] Biedenharn L C 1989 *J. Phys. A: Math. Gen.* **22** L873
[10] Macfarlane A J 1989 *J. Phys. A: Math. Gen.* **22** 4581
[11] Celeghini E, Rasetti M and Vitiello G 1991 *Phys. Rev. Lett.* **66** 2056
[12] Chang Z 1992 Minimum uncertainty states, two-photon device and quantum group symmetry *Preprint* BIHEP-TH-92-3
[13] Katriel J, Solomon A I, D'Ariano G and Rassetti M 1986 *Phys. Rev.* D **34** 2332
[14] Katriel J, Rassetti M and Solomon A I 1987 *Phys. Rev.* D **35** 1248